NON-AUTONOMOUS ORNSTEIN-UHLENBECK EQUATIONS IN EXTERIOR DOMAINS

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ABSTRACT. In this paper, we consider non-autonomous Ornstein-Uhlenbeck operators in smooth exterior domains $\Omega \subset \mathbb{R}^d$ subject to Dirichlet boundary conditions. Under suitable assumptions on the coefficients, the solution of the corresponding non-autonomous parabolic Cauchy problem is governed by an evolution system $\{P_\Omega(t,s)\}_{0\le s\le t}$ on $L^p(\Omega)$ for $1<p<\infty$. Furthermore, L^p-estimates for spatial derivatives and L^p-L^q smoothing properties of $P_\Omega(t,s)$, $0\le s\le t$, are obtained.

1. INTRODUCTION

In recent years, parabolic equations with unbounded and time-independent coefficients were investigated intensively in various function spaces over the whole space $\mathbb{R}^d$ or exterior domains; we refer e.g. to [6,8,9,13,15] and the monograph [5]. However, it is also interesting to consider parabolic equations with unbounded coefficients in the non-autonomous case. In particular, analytically there is a great interest in the prototype situation of time-dependent Ornstein-Uhlenbeck operators in exterior domains, as operators of this type arise e.g. in the study of the Navier-Stokes flow in the exterior of a rotating obstacle; see e.g. [12,16].

Therefore, in this paper we consider non-autonomous Cauchy problems with Dirichlet boundary condition of the type

$$\begin{cases} u_t(t,x) - \mathcal{L}_\Omega(t)u(t,x) = 0, & t\in(s,\infty),\ x\in\Omega,\\ u(t,x) = 0, & t\in(s,\infty),\ x\in\partial\Omega,\\ u(s,x) = f(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $s\ge 0$ is fixed, $\Omega\subset\mathbb{R}^d$ is a domain and $\{\mathcal{L}_\Omega(t)\}_{t\ge 0}$ is a family of time-dependent Ornstein-Uhlenbeck operators formally defined by

$$\mathcal{L}_\Omega(t)\varphi(x) = \frac{1}{2}\mathrm{Tr}\left(Q(t)Q^*(t)\mathrm{D}_x^2\varphi(x)\right) + \langle M(t)x + c(t), \mathrm{D}_x\varphi(x)\rangle, \quad x\in\Omega, \quad t\ge 0. \tag{1.2}$$

Throughout the paper we assume that Q, $M\in C^\alpha_{loc}(\mathbb{R}_+,\mathbb{R}^{d\times d})$, $c\in C^\alpha_{loc}(\mathbb{R}_+,\mathbb{R}^d)$ for some $\alpha\in(0,1)$ and there is $\mu>0$ such that

$$|Q(t)x|\ge\mu|x|, \quad t\ge 0,\ x\in\mathbb{R}^d.$$

The above assumption guaranties that the operators $\mathcal{L}_\Omega(t)$ are uniformly elliptic.

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The main purpose of this paper is to consider problem (1.1) in the L^p-setting for the case of smooth exterior domains Ω. However, in the course of this paper we also consider the situation where Ω is $\mathbb{R}^d$ and a smooth bounded domain.

In the following the L^p-realization of $\mathcal{L}_\Omega(t)$ will be denoted by $L_\Omega(t)$ with an appropriate domain $\mathcal{D}(L_\Omega(t)) \subset L^p(\Omega)$, specified later. Then we can rewrite equation (1.1) as an abstract non-autonomous Cauchy problem

$$\text{(nACP)} \qquad \begin{cases} u'(t) &= L_\Omega(t)u(t), \quad 0 \le s < t, \\ u(s) &= f, \end{cases} \tag{1.3}$$

where $f \in L^p(\Omega)$.

Definition 1.1. A continuous function $u : [s, \infty) \to L^p(\Omega)$ is called a *(classical) solution* of (nACP) if $u \in C^1((s,\infty), L^p(\Omega))$, $u(s) = f$, and $u'(t) = L_\Omega(t)u(t)$ for $0 \le s < t$.

Definition 1.2 (Well-posedness)**.** We say that the Cauchy problem (nACP) is *well-posed* (on *regularity spaces* $\{Y_s\}_{s\ge 0}$) if the following statements are true.

(i) **(Existence and uniqueness)** There are dense subspaces $Y_s \subset \mathcal{D}(L_\Omega(s))$ of $L^p(\Omega)$ such that for $f \in Y_s$ there is a unique solution $t \mapsto u(t; s, f) \in Y_t$ of (nACP).

(ii) **(Continuous dependence)** The solution depends continuously on the data; i.e., for $s_n \to s$ and $Y_{s_n} \ni f_n \to f \in Y_s$, we have $\tilde{u}(t; s_n, f_n) \to \tilde{u}(t; s, f)$ uniformly for t in compact subsets of $[0, \infty)$, where we set $\tilde{u}(t; s, f) := u(t; s, f)$ for $t \ge s$ and $\tilde{u}(t; s, f) := f$ for $t < s$.

In order to discuss well-posedness of (nACP) we introduce the concept of strongly continuous evolution systems.

Definition 1.3 (Evolution system)**.** A two parameter family of linear, bounded operators $\{P_\Omega(t,s)\}_{0\le s\le t}$ on $L^p(\Omega)$ is called a *(strongly continuous) evolution system* if

(i) $P_\Omega(s,s) = \mathrm{Id}$ and $P_\Omega(t,s) = P_\Omega(t,r)P_\Omega(r,s)$ for $0 \le s \le r \le t$,

(ii) for each $f \in L^p(\Omega)$, $(t,s) \mapsto P_\Omega(t,s)f$ is continuous on $0 \le s \le t$.

We say $\{P_\Omega(t,s)\}_{0\le s\le t}$ solves the Cauchy problem (nACP) (on spaces $\{Y_s\}_{s\ge 0}$) if there are dense subspaces Y_s of $L^p(\Omega)$ such that $P_\Omega(t,s)Y_s \subset Y_t \subset \mathcal{D}(L_\Omega(t))$ for $0 \le s \le t$ and the function $u(t) := P_\Omega(t,s)f$ is a solution of (nACP) for $f \in Y_s$.

It is well-known that the Cauchy problem (nACP) is well-posed on $\{Y_s\}_{s\ge 0}$ if and only if there is an evolution system solving (nACP) on $\{Y_s\}_{s\ge 0}$ (see e.g. [20, Sect. 3.2]).

The main result of this paper (see Theorem 3.1) is to show that for smooth exterior domains $\Omega \subset \mathbb{R}^d$ problem (nACP) is solved by a strongly continuous evolution system $\{P_\Omega(t,s)\}_{0\le s\le t}$ on $L^p(\Omega)$ and thus, is well-posed. Since in unbounded domains the operators $\mathcal{L}_\Omega(t)$ have unbounded drift coefficients, the present situation does not fit into the well-studied framework of evolution systems of parabolic type (see e.g. the monograph by Lunardi [17, Chapter 6] or the fundamental papers by Tanabe [22–24] and Acquistapace, Terreni [1–3]). Therefore the well-posedness of (nACP) and regularity properties of the solution do not follow from abstract arguments. Here lies the major difficulty. In order to prove our result we proceed as follows: In Section 2 we consider (nACP) in the case

that Ω is the whole space $\mathbb{R}^d$ or a smooth bounded domain. For the whole space case we use a representation formula for the evolution system as done in [7, 10]. In the case of bounded domains we can apply the standard results for non-autonomous Cauchy problems of parabolic type. These auxiliary results are then applied in Section 3 to construct an evolution system $\{P_\Omega(t,s)\}_{0\le s\le t}$ on $L^p(\Omega)$ for smooth exterior domains $\Omega \subset \mathbb{R}^d$, by some cut-off techniques. Moreover, our method allows us to prove L^p-L^q estimates and estimates for spatial derivatives of $\{P_\Omega(t,s)\}_{0\le s\le t}$.

Notations. The euclidian norm of $x \in \mathbb{R}^d$ will be denoted by $|x|$. By $B(R)$ we denote the open ball in $\mathbb{R}^d$ with centre at the origin and radius R. For $T > 0$ we use the notations:

$$\begin{aligned}
\Lambda_T &:= \{(t,s) : 0 \le s \le t \le T\}\\
\widetilde{\Lambda}_T &:= \{(t,s) : 0 \le s < t \le T\}\\
\Lambda &:= \{(t,s) : 0 \le s \le t\}\\
\widetilde{\Lambda} &:= \{(t,s) : 0 \le s < t\}.
\end{aligned}$$

If $u : \Omega \to \mathbb{R}$, where $\Omega \subseteq \mathbb{R}^d$ is a domain, we use the following notation:

$$\begin{aligned}
\mathrm{D}_i u &= \frac{\partial u}{\partial x_i},\ \mathrm{D}_{ij} u = \mathrm{D}_i \mathrm{D}_j u,\\
\mathrm{D}_x u &= (\mathrm{D}_1 u, \ldots, \mathrm{D}_d u),\ \mathrm{D}_x^2 u = (\mathrm{D}_{ij} u).
\end{aligned}$$

Let us come to notation for function spaces. For $1 \le p < \infty$, $j \in \mathbb{N}$, $W^{j,p}(\Omega)$ denotes the classical Sobolev space of all $L^p(\Omega)$–functions having weak derivatives in $L^p(\Omega)$ up to the order j. Its usual norm is denoted by $\|\cdot\|_{j,p}$ and by $\|\cdot\|_p$ when $j = 0$. By $W_0^{1,p}(\mathbb{R}^d)$ we denote the closure of the space of test functions $C_c^\infty(\mathbb{R}^d)$ with respect to the norm of $W^{1,p}(\mathbb{R}^d)$. For $0 < \alpha < 1$ we denote by $C^\alpha_{loc}(\mathbb{R}_+, \mathbb{R}^{d\times d})$ the space of all α-Hölder continuous functions in $[0,T]$ for all $T > 0$. The space of all bounded continuous functions $u : \Omega \to \mathbb{R}$ is denoted by $C_b(\Omega)$. For $k \in \mathbb{N}$, $C_b^k(\Omega)$ is the subspace of $C_b(\Omega)$ consisting of all functions which are differentiable up to the order k in Ω such that the derivatives are bounded. Finally, we denote by $C^{1,2}(I \times \Omega)$ the space of all functions $u : I \times \Omega \to \mathbb{R}$ which are continuously differentiable with respect to $t \in I$ and C^2 with respect to the space variable $x \in \Omega$, where $I \subseteq [0,\infty)$ is an interval.

2. Auxiliary results: whole space and bounded domains

In this section we prove some auxiliary results concerning the evolution systems in the case of the whole space $\mathbb{R}^d$ and smooth bounded domains. These results are needed in Section 3 for the construction of the evolution system in the case of exterior domains.

2.1. **The evolution system in the whole space.** The realizations of $\{\mathcal{L}_{\mathbb{R}^d}(t)\}_{t\ge 0}$ are defined by

$$\begin{aligned}
\mathcal{D}(L_{\mathbb{R}^d}(t)) &:= \{u \in W^{2,p}(\mathbb{R}^d) : \langle M(t)x, \mathrm{D}_x u(x)\rangle \in L^p(\mathbb{R}^d)\},\\
L_{\mathbb{R}^d}(t)u &:= \mathcal{L}_{\mathbb{R}^d}(t)u.
\end{aligned} \tag{2.1}$$

Here the domain of $L_\Omega(t)$ depends on the time parameter t. However, note that the subspace

$$Y_{\mathbb{R}^d} := \{u \in W^{2,p}(\mathbb{R}^d) : |x| \cdot \mathrm{D}_j u(x) \in L^p(\mathbb{R}^d) \text{ for all } j = 1, \dots, d\}$$

is contained in $\mathcal{D}(L_\Omega(t))$ for all $t \geq 0$ and is dense in $L^p(\mathbb{R}^d)$. The space $Y_{\mathbb{R}^d}$ will serve as a regularity space in order to discuss well-posedness of (nACP).

It follows directly from [19] (see also [18]) that in the autonomous case (i.e. for fixed $s \geq 0$) the operator $(L_{\mathbb{R}^d}(s), \mathcal{D}(L_{\mathbb{R}^d}(s))$ generates a strongly continuous semigroup, which is however not analytic. Second order elliptic operators in $\mathbb{R}^d$ with more general unbounded and time-independent coefficients were considered e.g. in [21], [14].

In the following we denote by $\{U(t,s)\}_{t,s\geq 0}$ the evolution system in $\mathbb{R}^d$ that satisfies

$$\begin{cases} \frac{\partial}{\partial t} U(t,s) &= -M(t)U(t,s), \\ U(s,s) &= \mathrm{Id}. \end{cases}$$

The existence of $\{U(t,s)\}_{t,s\geq 0}$ follows directly from the Picard-Lindelöf theorem. Now for $f \in L^p(\mathbb{R}^d)$ and $s \geq 0$ we set $P_{\mathbb{R}^d}(s,s) = \mathrm{Id}$ and for $(t,s) \in \widetilde{\Lambda}$ we define

$$P_{\mathbb{R}^d}(t,s)f(x) = (k(t,s,\cdot) * f)(U(s,t)x + g(t,s)), \qquad x \in \mathbb{R}^d, \tag{2.2}$$

where

$$k(t,s,x) := \frac{1}{(2\pi)^{\frac{d}{2}} (\det Q_{t,s})^{\frac{1}{2}}} \, \mathrm{e}^{-\frac{1}{2}\langle Q_{t,s}^{-1} x, x\rangle}, \qquad x \in \mathbb{R}^d, \tag{2.3}$$

$$g(t,s) = \int_s^t U(s,r)c(r)\mathrm{d}r \qquad \text{and} \qquad Q_{t,s} = \int_s^t U(s,r)Q(r)Q^*(r)U^*(s,r)\mathrm{d}r. \tag{2.4}$$

As in [7, Proposition 2.1] (see also [12, Proposition 2.1]) it can be shown that for initial value $f \in C_b^2(\mathbb{R}^d)$, the function $u(t,x) := P_{\mathbb{R}^d}(t,s)f(x)$ is a classical solution to

$$\begin{cases} u_t(t,x) - \mathcal{L}_{\mathbb{R}^d}(t)u(t,x) &= 0, \qquad (t,s) \in \widetilde{\Lambda},\ x \in \mathbb{R}^d, \\ u(s,x) &= f(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{2.5}$$

i.e. $u \in C^{1,2}((s,\infty)\times\Omega)$ and u solves (2.5). Further, the two parameter family of operators $\{P_{\mathbb{R}^d}(t,s)\}_{(t,s)\in\Lambda}$ is a strongly continuous evolution system on $L^p(\mathbb{R}^d)$.

Proposition 2.1. *Let $1 < p < \infty$. Then the family of operators $\{P_{\mathbb{R}^d}(t,s)\}_{(t,s)\in\Lambda}$ defined in (2.2) is a strongly continuous evolution system on $L^p(\mathbb{R}^d)$ with the following properties.*

(a) For $(t,s) \in \Lambda$, the operator $P_{\mathbb{R}^d}(t,s)$ maps $Y_{\mathbb{R}^d}$ into $Y_{\mathbb{R}^d}$.

(b) For every $f \in Y_{\mathbb{R}^d}$ and every $s \in [0,\infty)$, the map $t \mapsto P_{\mathbb{R}^d}(t,s)f$ is differentiable in (s,∞) and

$$\frac{\partial}{\partial t} P_{\mathbb{R}^d}(t,s)f = L_{\mathbb{R}^d}(t)P_{\mathbb{R}^d}(t,s)f. \tag{2.6}$$

(c) For every $f \in Y_{\mathbb{R}^d}$ and $t \in (0,\infty)$, the map $s \mapsto P_{\mathbb{R}^d}(t,s)f$ is differentiable in $[0,t)$ and

$$\frac{\partial}{\partial s} P_{\mathbb{R}^d}(t,s)f = -P_{\mathbb{R}^d}(t,s)L_{\mathbb{R}^d}(s)f. \tag{2.7}$$

Proof. In [10, Proposition 2.4] it was shown that the law of evolution (property (i) of Definition 1.3) holds for every $f \in C_c^\infty(\mathbb{R}^d)$. Since $C_c^\infty(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ the law of evolution holds even for all $f \in L^p(\mathbb{R}^d)$. The strong continuity of the map $\Lambda \ni (t,s) \mapsto P_{\mathbb{R}^d}(t,s)$ can be shown as in [12, Proposition 2.3]. Equalities (2.6) and (2.7) follow by differentiating the kernel $k(t,s,x)$ with respect to t and s, respectively.

Let us now show that the evolution system $\{P_{\mathbb{R}^d}(t,s)\}_{(t,s)\in\Lambda}$ leaves the regularity space $Y_{\mathbb{R}^d}$ invariant. Since $k(t,s,\cdot) \in C^\infty(\mathbb{R}^d)$ it follows that $P_{\mathbb{R}^d}(t,s)f \in C^\infty(\mathbb{R}^d)$ for all $f \in L^p(\mathbb{R}^d)$ and $(t,s) \in \widetilde{\Lambda}$. Moreover, we note that

$$\mathrm{D}_x P_{\mathbb{R}^d}(t,s)f = U^*(s,t)\left(k(t,s,\cdot) * \mathrm{D}_x f\right)\left(U(s,t)x + g(t,s)\right)$$

holds for all $f \in W^{1,p}(\mathbb{R}^d)$. Thus, it suffices to show that for all $j = 1,\dots,d$ we have $|x| \cdot \left(k(t,s,\cdot) * \mathrm{D}_j f\right)(x) \in L^p(\mathbb{R}^d)$. So let $h \in L^q(\mathbb{R}^d)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left| \left(|x| \cdot \left(k(t,s,\cdot) * \mathrm{D}_j f\right)(x)\right) h(x) \right| \mathrm{d}x \\
&\qquad \le C \int_{\mathbb{R}^d} |x||h(x)| \int_{\mathbb{R}^d} |\mathrm{D}_j f(x-y) \mathrm{e}^{-\frac{1}{2}\langle Q_{t,s}^{-1}y,y\rangle}| \mathrm{d}y \,\mathrm{d}x \\
&\qquad \le C \Bigg[\int_{\mathbb{R}^d} \mathrm{e}^{-\frac{1}{2}\langle Q_{t,s}^{-1}y,y\rangle} \int_{\mathbb{R}^d} \left| \left(|x-y| \cdot \mathrm{D}_j f(x-y)\right) h(x) \right| \mathrm{d}x \,\mathrm{d}y + \\
&\qquad\qquad \int_{\mathbb{R}^d} |y| \mathrm{e}^{-\frac{1}{2}\langle Q_{t,s}^{-1}y,y\rangle} \int_{\mathbb{R}^d} |\mathrm{D}_j f(x-y)||h(x)| \mathrm{d}x \,\mathrm{d}y \Bigg] \\
&\qquad \le C \left[\||x|\mathrm{D}_j f\|_p \|h\|_q + \|\mathrm{D}_j f\|_p \|h\|_q \right].
\end{aligned}$$

Here the constant C may change from line to line. Thus

$$\int_{\mathbb{R}^d} \left| \left(|x| \cdot \left(k(t,s,\cdot) * \mathrm{D}_j f\right)(x)\right) h(x) \right| \mathrm{d}x < \infty$$

holds for all $h \in L^q(\mathbb{R}^d)$ and this proves the assertion. $\square$

As a consequence of Proposition 2.1, Cauchy problem (nACP) is well-posed in the case of $\mathbb{R}^d$ with regularity space $Y_{\mathbb{R}^d}$. Now we prove L^p-L^q estimates and estimates for higher order spatial derivatives of $\{P_{\mathbb{R}^d}(t,s)\}_{(t,s)\in\Lambda}$. For this purpose we need the following estimates for the matrices $Q_{t,s}$. For a proof we refer to [10, Lemma 3.2] and [12, Lemma 2.4].

Lemma 2.2. *Let $T > 0$. Then there exists a constant $C := C(T) > 0$ such that*

$$\begin{aligned}
\|Q_{t,s}^{-\frac{1}{2}}\| &\le C(t-s)^{-\frac{1}{2}}, \quad (t,s) \in \widetilde{\Lambda}_T, \\
(\det Q_{t,s})^{\frac{1}{2}} &\ge C(t-s)^{\frac{d}{2}}, \quad (t,s) \in \Lambda_T.
\end{aligned} \tag{2.8}$$

Proposition 2.3. *Let $T > 0$, $1 < p \le q < \infty$ and $\beta \in \mathbb{N}_0^d$ be a multi-index . Then there exists a constant $C := C(T) > 0$ such that for every $f \in L^p(\mathbb{R}^d)$*

(a) $\|P_{\mathbb{R}^d}(t,s)f\|_q \le C(t-s)^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_p, \qquad (t,s) \in \widetilde{\Lambda}_T,$

(b) $\|\mathrm{D}_x^\beta P_{\mathbb{R}^d}(t,s)f\|_p \le C(t-s)^{-\frac{|\beta|}{2}}\|f\|_p, \qquad (t,s) \in \widetilde{\Lambda}_T.$

Moreover,

$$\|P_{\mathbb{R}^d}(t,s)f\|_{k,p} \le C\|f\|_{k,p}, \quad (t,s)\in\Lambda_T,$$

for all $f\in W^{k,p}(\mathbb{R}^d)$, $k=1,\,2$, *and*

$$\|P_{\mathbb{R}^d}(t,s)f\|_{2,p} \le C(t-s)^{-\frac{1}{2}}\|f\|_{1,p}, \quad (t,s)\in\widetilde{\Lambda}_T,$$

for all $f\in W^{1,p}(\mathbb{R}^d)$.

Proof. Let $T>0$. By the change of variables $\xi = U(s,t)x$ and by Young's inequality we obtain

$$\|P_{\mathbb{R}^d}(t,s)f\|_q \le |\det U(s,t)|^{\frac{1}{q}}\|k(t,s,\cdot)\|_r\|f\|_p,$$

where $1<r<\infty$ with $\frac{1}{p}+\frac{1}{r}=1+\frac{1}{q}$. Moreover, by the change of variables $y=Q_{t,s}^{1/2}z$ we obtain

$$\|k(t,s,\cdot)\|_r^r = \frac{(\det Q_{t,s})^{\frac{1}{2}(1-r)}}{(2\pi)^{\frac{d}{2}\cdot r}}\int_{\mathbb{R}^d} \mathrm{e}^{-\frac{r|z|^2}{2}}\mathrm{d}z \le C(\det Q_{t,s})^{\frac{1}{2}(1-r)}.$$

Now Lemma 2.2 yields (a).

To prove (b) we first note that

$$\left|\mathrm{D}_x^\beta P_{\mathbb{R}^d}(t,s)f(x)\right| \le |U^*(s,t)|^{|\beta|}\left|\left(\mathrm{D}_x^\beta k(t,s,\cdot)*f\right)(U(s,t)x+g(t,s))\right|$$

holds. Thus, we have to estimate the norm of $\mathrm{D}_x^\beta k(t,s,\cdot)$. Since

$$\mathrm{D}_x k(t,s,x) = -k(t,s,x)\left(Q_{t,s}^{-1}x\right)^*$$

holds, we obtain by differentiating further

$$|\mathrm{D}_x^\beta k(t,s,x)| \le Ck(t,s,x)|Q_{t,s}^{-1}x|^{|\beta|}$$

for some constant $C>0$. As above, by the change of variables $y=Q_{t,s}^{1/2}z$, we obtain

$$\|\mathrm{D}_x^\beta k(t,s,\cdot)\|_1 \le \frac{\|Q_{t,s}^{-\frac{1}{2}}\|^{|\beta|}}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}|z|^{|\beta|}\mathrm{e}^{-\frac{|z|^2}{2}}\mathrm{d}z \le C\|Q_{t,s}^{-\frac{1}{2}}\|^{|\beta|}.$$

Now Lemma 2.2 yields assertion (b). The last assertions follow by a direct computation. □

Remark 2.4. If $\{U(t,s)\}_{t,s\ge 0}$ is uniformly bounded, i.e. $\|U(t,s)\|\le M$ for some constant $M>0$ and all $t,s\ge 0$, then the estimates in Lemma 2.2 and Proposition 2.3 hold in Λ and $\widetilde{\Lambda}$ respectively. In particular, in this case the evolution system $\{P(t,s)\}_{(t,s)\in\Lambda}$ is uniformly bounded.

2.2. **The evolution system in bounded domains.** In this subsection we assume that $D\subset\mathbb{R}^d$ is a bounded domain with $C^{1,1}$-boundary. For $t\ge 0$ we set

$$\begin{aligned} \mathcal{D}(L_D(t)) &=: \mathcal{D}(L_D) := W^{2,p}(D)\cap W_0^{1,p}(D),\\ L_D(t)u &:= \mathcal{L}_D(t)u. \end{aligned} \tag{2.9}$$

Note that in this situation the domain is independent of the time parameter t, i.e. all the operators $L_D(t)$ are defined on the same domain $\mathcal{D}(L_D)$.

Lemma 2.5. *Let $D \subset \mathbb{R}^d$ be a bounded domain with $C^{1,1}$-boundary and $1 < p < \infty$.*

(a) For fixed $s \in [0, \infty)$, the operator $(L_D(s), \mathcal{D}(L_D))$ generates an analytic semigroup on $L^p(D)$.

(b) The map $t \mapsto L_D(t)$ belongs to $C^{\alpha}_{loc}(\mathbb{R}_+, \mathscr{L}(\mathcal{D}(L_D), L^p(D)))$.

Proof. Assertion (a) follows from the classical theory of elliptic second order operators in bounded domains (see also [9, Lemma 2.4]). Assertion (b) follows from the assumptions on the coefficients of $L_D(\cdot)$. □

The following proposition now follows directly from the theory of evolution systems of parabolic type; see [17, Chapter 6] and [11, Sect. 2.3]. See also [4, Sect. 7] for bounded domains of class C^2.

Proposition 2.6. *Let $D \subset \mathbb{R}^d$ be a bounded domain with $C^{1,1}$-boundary and $1 < p < \infty$. Then there is a unique evolution system $\{P_D(t,s)\}_{(t,s)\in\Lambda}$ on $L^p(D)$ with the following properties.*

(a) For $(t,s) \in \widetilde{\Lambda}$, the operator $P_D(t,s)$ maps $L^p(D)$ into $\mathcal{D}(L_D)$.

(b) The map $t \mapsto P_D(t,s)$ is differentiable in (s,∞) with values in $\mathscr{L}(L^p(D))$ and

$$\frac{\partial}{\partial t} P_D(t,s) = L_D(t) P_D(t,s). \tag{2.10}$$

(c) For every $f \in \mathcal{D}(L_D)$ and $t \in (0,\infty)$, the map $s \mapsto P_D(t,s)f$ is differentiable in $[0,t)$ and

$$\frac{\partial}{\partial s} P_D(t,s)f = -P_D(t,s) L_D(s) f. \tag{2.11}$$

(d) Let $T > 0$. Then there exists a constant $C := C(T) > 0$ such that

$$\|P_D(t,s)f\|_p \le C \|f\|_p, \tag{2.12}$$

and

$$\|P_D(t,s)f\|_{2,p} \le C(t-s)^{-1} \|f\|_p. \tag{2.13}$$

for all $f \in L^p(D)$ and all $(t,s) \in \widetilde{\Lambda}_T$.

The following estimates follow directly from the proposition above and simple interpolation.

Corollary 2.7. *Let $T > 0$, $1 < p < \infty$ and $p \le q < \infty$. Then there exists a constant $C := C(T) > 0$ such that for every $f \in L^p(D)$*

(a) $\|P_D(t,s)f\|_q \le C(t-s)^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_p, \qquad (t,s) \in \widetilde{\Lambda}_T,$

(b) $\|\mathrm{D}_x P_D(t,s)f\|_p \le C(t-s)^{-\frac{1}{2}} \|f\|_p, \qquad (t,s) \in \widetilde{\Lambda}_T.$

Moreover,

$$\|P_D(t,s)f\|_{k,p} \le C \|f\|_{k,p}, \quad (t,s) \in \Lambda_T,$$

for all $f \in W^{k,p}(D)$, $k = 1, 2$, and

$$\|P_D(t,s)f\|_{2,p} \le C(t-s)^{-\frac{1}{2}} \|f\|_{1,p}, \quad (t,s) \in \widetilde{\Lambda}_T,$$

for all $f \in W^{1,p}(D)$.

Proof. Let us start with the case $q \geq p \geq d/2$. Then, by the Gagliardo-Nierenberg inequality (cf. [25, Theorem 3.3]) and Proposition 2.6 (d), we immediately obtain

$$\|P_D(t,s)f\|_q \leq C\|\mathrm{D}_x^2 P_D(t,s)f\|_p^a \|P_D(t,s)f\|_p^{1-a} \leq C(t-s)^{-a}\|f\|_p,\ (t,s) \in \widetilde{\Lambda}_T,$$

where $a = \frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)$. The case $1 < p < \frac{d}{2}$ follows by iteration. Assertion (b) is also proved by the Gagliardo-Nierenberg inequality. By setting $a = \frac{1}{2}$ and $p = q$ we obtain

$$\|\mathrm{D}_x P_D(t,s)f\|_p \leq C\|\mathrm{D}_x^2 P_D(t,s)f\|_p^{\frac{1}{2}} \|P_D(t,s)f\|_p^{\frac{1}{2}} \leq C(t-s)^{-\frac{1}{2}}\|f\|_p,\ (t,s) \in \widetilde{\Lambda}_T.$$

For the last assertions we refer, for example, to [17, Corollary 6.1.8]. □

3. The evolution system in exterior domains

In this section we come to the main part of this paper. In the sequel we always assume that $\Omega \subset \mathbb{R}^d$ is an exterior domain with $C^{1,1}$-boundary, i.e., $\Omega = \mathbb{R}^d \setminus K$, where $K \subset \mathbb{R}^d$ is a compact set with $C^{1,1}$-boundary. For $t \geq 0$ we set

$$\begin{aligned} \mathcal{D}(L_\Omega(t)) &:= \{u \in W^{2,p}(\mathbb{R}^d) \cap W_0^{1,p}(\Omega) : \langle M(t)x, \mathrm{D}_x u(x)\rangle \in L^p(\Omega)\}, \\ L_\Omega(t)u &:= \mathcal{L}_\Omega(t)u. \end{aligned} \tag{3.1}$$

Here the domain of $L_\Omega(t)$ depends on the time parameter t, however the subspace

$$Y_\Omega := \{u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) : |x| \cdot \mathrm{D}_j u(x) \in L^p(\Omega) \text{ for } j = 1, \ldots, d\}$$

is contained in $\mathcal{D}(L_\Omega(t))$ for all $t \geq 0$ and is dense in $L^p(\Omega)$. It follows from [9] that in the autonomous case (i.e. for fixed $s \geq 0$) the operator $(L_\Omega(s), \mathcal{D}(L_\Omega(s))$ generates a strongly continuous semigroup on $L^p(\Omega)$. For more general second order elliptic operators with unbounded and time-independent coefficients in exterior domains we refer to [13]. Our main result is the existence of an evolution system in $L^p(\Omega)$, $1 < p < \infty$, associated to the operators $L_\Omega(\cdot)$.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^d$ be an exterior domain with $C^{1,1}$-boundary and $1 < p < \infty$. Then there exists a unique evolution system $\{P_\Omega(t,s)\}_{(t,s)\in\Lambda}$ on $L^p(\Omega)$ with the following properties.*

(a) For $(t,s) \in \Lambda$, the operator $P_\Omega(t,s)$ maps Y_Ω into Y_Ω.

(b) For every $f \in Y_\Omega$ and $s \geq 0$, the map $t \mapsto P_\Omega(t,s)f$ is differentiable in (s,∞) and

$$\frac{\partial}{\partial t} P_\Omega(t,s)f = L_\Omega(t)P_\Omega(t,s)f. \tag{3.2}$$

(c) For every $f \in Y_\Omega$ and $t > 0$, the map $s \mapsto P_\Omega(t,s)f$ is differentiable in $[0,t)$ and

$$\frac{\partial}{\partial s} P_\Omega(t,s)f = -P_\Omega(t,s)L_\Omega(s)f. \tag{3.3}$$

As a direct consequence we obtain well-posedness of the abstract non-autonomous Cauchy problem (nACP) on the regularity space Y_Ω.

Corollary 3.2. *Let Ω be an exterior $C^{1,1}$-domain. Then the Cauchy problem (nACP) is well-posed on Y_Ω.*

In the following, we describe the construction of the evolution system $\{P_\Omega(t,s)\}_{(t,s)\in\Lambda}$ in detail. The general idea is to derive the result for exterior domains from the corresponding results in the case of $\mathbb{R}^d$ and bounded domains. For this purpose let $R>0$ be such that $K\subset B(R)$. We set $D:=\Omega\cap B(R+3)$. We denote by $\{P_{\mathbb{R}^d}(t,s)\}_{(t,s)\in\Lambda}$ the evolution system in $L^p(\mathbb{R}^d)$ and by $\{P_D(t,s)\}_{(t,s)\in\Lambda}$ the evolution system in $L^p(D)$ for the bounded domain D. Next we choose cut-off functions $\varphi,\eta\in C^\infty(\Omega)$ such that $0\le\varphi,\eta\le 1$ and

$$\varphi(x):=\begin{cases}1, & |x|\ge R+2,\\ 0, & |x|\le R+1,\end{cases}$$

and

$$\eta(x):=\begin{cases}1, & |x|\le R+2,\\ 0, & |x|\ge R+\frac{5}{2}.\end{cases}$$

For $f\in L^p(\Omega)$ we define $f_0\in L^p(\mathbb{R}^d)$ and $f_D\in L^p(D)$, respectively, by

$$f_0(x):=\begin{cases}f(x), & x\in\Omega,\\ 0, & x\notin\Omega,\end{cases}\qquad\text{and}\qquad f_D(x)=\eta(x)f(x).$$

These definitions ensure that for every function $f\in\mathcal{D}(L_\Omega(t))$ we have $f_0\in\mathcal{D}(L_{\mathbb{R}^d}(t))$ and $f_D\in\mathcal{D}(L_D(t))$. Now for $(t,s)\in\Lambda$ and $f\in L^p(\Omega)$ we set

$$W(t,s)f=\varphi P_{\mathbb{R}^d}(t,s)f_0+(1-\varphi)P_D(t,s)f_D. \tag{3.4}$$

A short calculation yields

$$\begin{aligned}\mathrm{D}_xW(t,s)f&=\varphi\mathrm{D}_xP_{\mathbb{R}^d}(t,s)f_0+(1-\varphi)\mathrm{D}_xP_D(t,s)f_D\\&\quad+\mathrm{D}_x\varphi\left(P_{\mathbb{R}^d}(t,s)f_0-P_D(t,s)f_D\right),\end{aligned}$$

and

$$\begin{aligned}\mathrm{D}_x^2W(t,s)f&=\varphi\mathrm{D}_x^2P_{\mathbb{R}^d}(t,s)f_0+(1-\varphi)\mathrm{D}_x^2P_D(t,s)f_D\\&\quad+2\left(\mathrm{D}_x\varphi\right)^*\cdot\left(\mathrm{D}_xP_{\mathbb{R}^d}(t,s)f_0-\mathrm{D}_xP_D(t,s)f_D\right)\\&\quad+\mathrm{D}_x^2\varphi\left(P_{\mathbb{R}^d}(t,s)f_0-P_D(t,s)f_D\right).\end{aligned}$$

Thus, for $f\in Y_\Omega$, we obtain

$$\begin{cases}\frac{\partial}{\partial t}W(t,s)f &= L_\Omega(t)W(t,s)f-F(t,s)f,\quad (t,s)\in\Lambda,\\ W(s,s)f &= f,\end{cases} \tag{3.5}$$

with

$$\begin{aligned}F(t,s)f&=\mathrm{Tr}\left[Q(t)Q^*(t)\left(\mathrm{D}_x\varphi\right)^*\cdot\left(\mathrm{D}_xP_{\mathbb{R}^d}(t,s)f_0-\mathrm{D}_xP_D(t,s)f_D\right)\right]\\&\quad+\mathcal{L}_\Omega(t)\varphi\left(P_{\mathbb{R}^d}(t,s)f_0-P_D(t,s)f_D\right).\end{aligned} \tag{3.6}$$

From the properties of the evolution systems $\{P_{\mathbb{R}^d}(t,s)\}_{(t,s)\in\Lambda}$ and $\{P_D(t,s)\}_{(t,s)\in\Lambda}$ it follows that the function $F(t,s)f$ in (3.6) is well-defined for every $f\in L^p(\Omega)$ and $(t,s)\in\widetilde{\Lambda}$.

Moreover, for every $f \in L^p(\Omega)$, $F(\cdot,\cdot)f$ is continuous in $\widetilde{\Lambda}$ with values in $L^p(\Omega)$. By using Proposition 2.3 and Corollary 2.7 we obtain the estimate

$$\|F(t,s)f\|_p \le C\left(1+(t-s)^{-\frac{1}{2}}\right)\|f\|_p, \qquad (t,s)\in\widetilde{\Lambda}_T, \tag{3.7}$$

for any $T>0$ and a suitable constant $C := C(T) > 0$.

It is clear, that if an evolution system $\{P_\Omega(t,s)\}_{(t,s)\in\Lambda}$ exists on $L^p(\Omega)$, then the solution $u(t)$ to the inhomogeneous equation (3.5) is given by the variation of constant formula

$$u(t) = P_\Omega(t,s)f - \int_s^t P_\Omega(t,r)F(r,s)f\,\mathrm{d}r.$$

This consideration suggests to consider the integral equation

$$P_\Omega(t,s)f = W(t,s)f + \int_s^t P_\Omega(t,r)F(r,s)f\,dr, \quad (t,s)\in\Lambda,\ f\in L^p(\Omega). \tag{3.8}$$

Let us state a lemma which will be very useful. Its proof is analogous to the proof in the case of one-parameter families (see [8, Lemma 4.6]). But for the sake of completeness we give here the details of the proof.

Lemma 3.3. *Let X_1 and X_2 be two Banach spaces, $T>0$ and let $R:\widetilde{\Lambda}_T\to\mathscr{L}(X_2,X_1)$ and $S:\widetilde{\Lambda}_T\to\mathscr{L}(X_2)$ be strongly continuous functions. Assume that*

$$\|R(t,s)\|_{\mathscr{L}(X_2,X_1)} \le C_0(t-s)^\alpha, \quad \|S(t,s)\|_{\mathscr{L}(X_2)} \le C_0(t-s)^\beta, \quad (t,s)\in\widetilde{\Lambda}_T,$$

holds for some $C_0 := C_0(T) > 0$ and $\alpha,\beta > -1$. For $f\in X_2$ and $(t,s)\in\widetilde{\Lambda}_T$, set $T_0(t,s)f := R(t,s)f$ and

$$T_n(t,s)f := \int_s^t T_{n-1}(t,r)S(r,s)f\,\mathrm{d}s, \qquad n\in\mathbb{N},\ (t,s)\in\widetilde{\Lambda}_T.$$

Then there exists a constant $C>0$ such that

$$\sum_{n=0}^\infty \|T_n(t,s)f\|_{X_1} \le C(t-s)^\alpha\|f\|_{X_2}, \qquad (t,s)\in\widetilde{\Lambda}_T. \tag{3.9}$$

Moreover, if $\alpha\ge 0$, the convergence of the series in (3.9) is uniform on Λ_T.

Proof. For $f\in X_2$ and $(t,s)\in\widetilde{\Lambda}_T$ we have

$$\|T_1(t,s)f\|_{X_1} \le C_0^2\int_s^t (t-r)^\alpha(r-s)^\beta\,dr = C_0^2(t-s)^{\alpha+\beta+1}\mathrm{B}(\beta+1,\alpha+1)\|f\|_{X_2},$$

where $\mathrm{B}(\cdot,\cdot)$ denotes the Beta function. So, by induction, we obtain

$$\begin{aligned}
&\|T_n(t,s)f\|_{X_1}\\
\le\ & C_0^{n+1}(t-s)^{\alpha+n(\beta+1)}\mathrm{B}(\beta+1,\alpha+1)\cdots\mathrm{B}(\beta+1,\alpha+1+(n-1)(\beta+1))\|f\|_{X_2}\\
=\ & C_0^{n+1}(t-s)^{\alpha+n(\beta+1)}\Gamma(\beta+1)^n\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1+n(\beta+1))}\|f\|_{X_2},\ n\in\mathbb{N},\ (t,s)\in\widetilde{\Lambda}_T,
\end{aligned}$$

where Γ denotes the Gamma function. Let us recall now the identity $\Gamma(x+1) = x\Gamma(x)$, $x > -1$, and denotes by $[\cdot]$ the Gaussian brackets. Then, it follows that

$$\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1+n(\beta+1))} \le \frac{C_\alpha}{[n(\beta+1)]!}, \quad n \in \mathbb{N}$$

for some $C_\alpha > 0$. Hence,

$$\begin{aligned} \|T_n(t,s)f\|_{X_1} &\le C_\alpha C_0 (t-s)^\alpha \Gamma(\beta+1)^n C_0^n \frac{(t-s)^{n(\beta+1)}}{[n(\beta+1)]!} \|f\|_{X_2} \\ &\le C_\alpha C_0 (t-s)^\alpha e^{t-s} \left(C_0 \Gamma(\beta+1)\right)^n \frac{(t-s)^{[n(\beta+1)]}}{[n(\beta+1)]!} \|f\|_{X_2},\, n \in \mathbb{N},\, (t,s) \in \widetilde{\Lambda}_T. \end{aligned}$$

Since

$$\begin{aligned} \sum_{n=0}^{\infty} \left(C_0 \Gamma(\beta+1)\right)^n \frac{(t-s)^{[n(\beta+1)]}}{[n(\beta+1)]!} &\le C_\beta e^{c_\beta (t-s)} \\ &\le C_\beta e^{c_\beta T} =: C_T, \quad (t,s) \in \Lambda_T \end{aligned}$$

for some constants C_β, $c_\beta > 0$, it follows that

$$\sum_{n=0}^{\infty} \|T_n(t,s)f\|_{X_1} \le C_T C_0 C_\alpha e^T t - s)^\alpha \|f\|_{X_2}, \quad (t,s) \in \widetilde{\Lambda}_T.$$

It is clear that if $\alpha \ge 0$ then the convergence of the above series is uniform on Λ_T. $\square$

Proof of Theorem 3.1. Let $T > 0$. By using Proposition 2.3 and Corollary 2.7 we have

$$\|W(t,s)f\|_p \le C\|f\|_p, \qquad \text{for } f \in L^p(\Omega),\ (t,s) \in \Lambda_T.$$

So, by (3.7), we can apply Lemma 3.3 with $R = W$, $S = F$, $\alpha = 0$, $\beta = -\frac{1}{2}$ and $X_1 = X_2 = L^p(\Omega)$. Thus, for any $f \in L^p(\Omega)$, the series $\sum_{k=0}^{\infty} P_k(t,s)f$ converges uniformly in Λ_T, where $P_0(t,s)f = W(t,s)f$ and

$$P_{k+1}(t,s)f = \int_s^t P_k(t,r)F(r,s)f\,\mathrm{d}r, \quad (t,s) \in \Lambda_T,\ f \in L^p(\mathbb{R}^d). \tag{3.10}$$

Since $T > 0$ is arbitrary,

$$P_\Omega(t,s) := \sum_{k=0}^{\infty} P_k(t,s), \quad (t,s) \in \Lambda \tag{3.11}$$

is well-defined. It is easy to check that $P_\Omega(t,s)$ satisfies the integral equation (3.8). Moreover, from the strong continuity of $W(\cdot,\cdot)$ and (3.7) we deduce inductively that $P_k(\cdot,\cdot)$ is strongly continuous and hence, by the uniform convergence of the series we get the strong continuity of $P_\Omega(\cdot,\cdot)$.

In order to show that $\{P_\Omega(t,s)\}_{(t,s)\in\Lambda}$ leaves Y_Ω invariant, we consider the Banach space $X_1 := \{f \in W_0^{1,p}(\Omega) : |x| \cdot \mathrm{D}_j f(x) \in L^p(\Omega) \text{ for } j = 1, \dots, d\}$ endowed with the norm

$$\|f\|_{X_1} := \|f\|_{1,p} + \||x| \cdot \mathrm{D}_x f\|_p, \quad f \in X_1.$$

Proposition 2.3, Corollary 2.7 and the last part of the proof of Proposition 2.1 permit us to apply Lemma 3.3 with $X_2 = X_1$, $R = W$, $S = F, \alpha = 0$ and $\beta = -\frac{1}{2}$. So, we obtain that $P_\Omega(t,s)f \in X_1$ for all $f \in X_1$ and $(t,s) \in \Lambda$. Moreover, by taking $X_1 = W^{2,p}(\Omega)$, $X_2 = W^{1,p}(\Omega)$, $R = W$, $S = F$, $\alpha = \beta = -\frac{1}{2}$ and applying Proposition 2.3 and Corollary 2.7, it follows, by Lemma 3.3, that $P_\Omega(t,s)f \in W^{2,p}(\Omega)$ for all $f \in W^{1,p}(\Omega)$ and $(t,s) \in \widetilde{\Lambda}$. This yields that $\{P_\Omega(t,s)\}_{(t,s)\in\Lambda}$ leaves Y_Ω invariant and

$$\begin{aligned}
&\sum_{n=0}^{\infty} [\|P_k(t,s)f\|_{2,p} + \||x|\mathrm{D}_x P_k(t,s)f\|_p] \\
&\qquad < C_T(1+(t-s)^{-\frac{1}{2}})(\|f\|_{1,p} + \||x|\cdot\mathrm{D}_x f\|_p), \quad (t,s) \in \widetilde{\Lambda}_T,\ f \in Y_\Omega. \qquad (3.12)
\end{aligned}$$

Let us now prove Equation (3.2). For $f \in Y_\Omega$ we compute

$$\begin{aligned}
\frac{\partial}{\partial t} P_0(t,s)f &= L_\Omega(t)P_0(t,s)f - F(t,s)f \\
\frac{\partial}{\partial t} P_1(t,s)f &= L_\Omega(t)P_1(t,s)f + F(t,s)f - \int_s^t F(t,r)F(r,s)f\mathrm{d}r \\
\frac{\partial}{\partial t} P_2(t,s)f &= L_\Omega(t)P_2(t,s)f + \int_s^t F(t,r)F(r,s)f\mathrm{d}r \\
&\qquad - \int_s^t \int_{r_1}^t F(t,r_2)F(r_2,r_1)F(r_1,s)f\mathrm{d}r_2\mathrm{d}r_1.
\end{aligned}$$

Inductively we see that

$$\frac{\partial}{\partial t} \sum_{k=0}^{n} P_k(t,s)f = L_\Omega(t) \sum_{k=0}^{n} P_k(t,s)f - R_n(t,s)f \qquad (3.13)$$

holds for $n \in \mathbb{N}$, where

$$R_n(t,s)f := \int_s^t \int_{r_1}^t \dots \int_{r_{n-1}}^t F(t,r_n)F(r_n,r_{n-1})\dots F(r_1,s)f\mathrm{d}r_n \dots \mathrm{d}r_2\mathrm{d}r_1.$$

Now, we estimate the norm of $R_n(t,s)f$. Estimate (3.6) yields

$$\begin{aligned}
\|R_1(t,s)f\|_p &\le C^2 \int_s^t (t-r)^{-\frac{1}{2}}(r-s)^{-\frac{1}{2}}\mathrm{d}r\|f\|_p = C^2\mathrm{B}(1/2,1/2)\|f\|_p, \\
\|R_2(t,s)f\|_p &\le C^3\mathrm{B}(1/2,1/2) \int_s^t (r-s)^{-\frac{1}{2}}\mathrm{d}r\|f\|_p \\
&= C^3\mathrm{B}(1/2,1/2)\mathrm{B}(1/2,1)(t-s)^{\frac{1}{2}}\|f\|_p.
\end{aligned}$$

Inductively, we see that

$$\begin{aligned}
\|R_n(t,s)\|_p &\le C^{n+1}\mathrm{B}(1/2,1/2)\mathrm{B}(1/2,1)\dots\mathrm{B}(1/2,n/2)(t-s)^{\frac{n-1}{2}}\|f\|_p \\
&\le \frac{C^{n+1}\Gamma(1/2)^n}{\left[\frac{n-1}{2}\right]!}(t-s)^{\frac{n-1}{2}}\|f\|_p \qquad (3.14)
\end{aligned}$$

holds for $n \in \mathbb{N}$. Here the constant C may change from line to line. From estimate (3.14) it follows that $\|R_n\|_p$ tends to zero as $n \to \infty$. So, by (3.12) and the closedness of $L_\Omega(t)$, we can conclude that

$$\frac{\partial}{\partial t} P_\Omega(t,s)f = L_\Omega(t) \sum_{k=0}^{\infty} P_k(t,s)f, \quad t > s, \ f \in Y_\Omega,$$

holds and this proves (3.2).

Let us now show Equation (3.3). For $f \in Y_\Omega$ we have

$$L_D(s)(\eta f) = \eta L_\Omega(s)f + \mathrm{Tr}[Q(t)Q^*(t)(\mathrm{D}_x\eta)^* \cdot \mathrm{D}_x f] + (\mathcal{L}_\Omega(s)\eta)f$$

holds. Thus,

$$\begin{aligned} W(t,s)L_\Omega(s)f &= \varphi P_{\mathbb{R}^d}(t,s)(L_\Omega(s)f)_0 + (1-\varphi)P_D(t,s)(L_\Omega(s)f)_D \\ &= \varphi P_{\mathbb{R}^d}(t,s)L_{\mathbb{R}^d}(s)f_0 + (1-\varphi)P_D(t,s)L_D(s)f_D - G(t,s)f, \end{aligned}$$

where

$$G(t,s)f := (1-\varphi)P_D(t,s)\left(\mathrm{Tr}[Q(t)Q^*(t)(\mathrm{D}_x\eta)^* \cdot \mathrm{D}_x f] + (\mathcal{L}_\Omega(s)\eta)f\right)$$

and $f \in Y_\Omega$. This yields

$$\frac{\partial}{\partial s} W(t,s)f = -W(t,s)L_\Omega(s)f - G(t,s)f$$

for $(t,s) \in \Lambda$ and $f \in Y_\Omega$.

Now, let $T > 0$ be arbitrary but fixed. Then, from the definition of G and Corollary 2.7, it follows that we can apply Lemma 3.3 with $X_1 = X_2 = W^{1,p}(\Omega)$, $R = S = G$ and $\alpha = \beta = -\frac{1}{2}$. So, the series

$$T(t,s)f := \sum_{k=0}^{\infty} T_k(t,s)f, \qquad (t,s) \in \widetilde{\Lambda}_T,$$

is well-defined and

$$\|T(t,s)f\|_{1,p} \le C(t-s)^{-\frac{1}{2}} \|f\|_{1,p}, \quad (t,s) \in \widetilde{\Lambda}_T, \tag{3.15}$$

for $f \in W^{1,p}(\Omega)$. On the other hand, $T(\cdot,\cdot)$ satisfies the integral equation

$$T(t,s)f = G(t,s)f + \int_s^t T(t,r)G(r,s)f \,\mathrm{d}r, \qquad (t,s) \in \Lambda_T, \ f \in W^{1,p}(\Omega). \tag{3.16}$$

In particular $T(t,\cdot)f$ is continuous on $[0,t]$ with respect to the L^p-norm for any $f \in W^{1,p}(\Omega)$ and $t \ge 0$. Now, for $f \in L^p(\Omega)$ and $(t,s) \in \Lambda_T$ we set

$$S(t,s)f := W(t,s)f + \int_s^t T(t,r)W(r,s)f \,\mathrm{d}r.$$

It follows from the continuity of $T(t,\cdot)W(\cdot,s)f$ on $[s,t]$, Proposition 2.3 and Corollary 2.7 that the above integral is well-defined for any $f \in L^p(\Omega)$. Computing the derivative with

respect to s yields

$$\begin{aligned}\frac{\partial}{\partial s}S(t,s)f &= -W(t,s)L_\Omega(s)f - G(t,s)f + T(t,s)f - \int_s^t T(t,r)W(r,s)L_\Omega(s)f\mathrm{d}r \\ &\quad - \int_s^t T(t,r)G(r,s)f\mathrm{d}r \\ &= -S(t,s)L_\Omega(s)f,\end{aligned}$$

for $f \in Y_\Omega$, due to (3.16). From this equality together with (3.2) and since $P_\Omega(t,s)Y_\Omega \subset Y_\Omega$, $(t,s) \in \Lambda$, we can conclude that

$$\frac{\partial}{\partial r}(S(t,r)P_\Omega(r,s)f) = 0$$

holds for all $f \in Y_\Omega$. This yields that for $f \in Y_\Omega$, the function $S(t,r)P_\Omega(r,s)f$ is constant on Λ_T and thus, by the density of Y_Ω in $L^p(\Omega)$ and by the fact that $T > 0$ was arbitrary, it follows that $S(t,s)f = P_\Omega(t,s)f$ holds for all $f \in L^p(\Omega)$ and all $(t,s) \in \Lambda$. This proves (3.3).

Let us now show the uniqueness of the solution $P_\Omega(t,s)f$ of (nACP) for initial value $f \in Y_\Omega$. For this purpose we assume that there exists another solution $t \mapsto u(t) \in Y_\Omega$. Since $u(r) \in Y_\Omega$ for all $r \in [s,\infty)$ it follows from equality (3.3) that the map $r \mapsto P_\Omega(t,r)u(r)$ is differentiable for $0 \le s < r < t$ and

$$\frac{\partial}{\partial r}\left(P_\Omega(t,r)u(r)\right) = -P_\Omega(t,r)L_\Omega(r)u(r) + P_\Omega(t,r)L_\Omega(r)u(r) = 0.$$

Therefore $P_\Omega(t,r)u(r)$ is constant on $0 \le s < r < t$. Thus, by letting $r \to s$ and $r \to t$ we obtain $P_\Omega(t,s)f = u(t)$. The uniqueness now directly implies that the law of evolution (Property (i) of Definition 1.3) holds. ☐

To conclude this section we prove L^p-L^q smoothing properties of the evolution system $\{P_\Omega(t,s)\}_{(t,s)\in\Lambda}$ and L^p-estimates for its spatial derivatives. The following estimates follow basically directly via the representation (3.11) from Lemma 3.3, Proposition 2.3 and Corollary 2.7.

Proposition 3.4. *Let $T > 0$, $1 < p < \infty$ and $p \le q < \infty$. Then there exists a constant $C := C(T) > 0$ such that*

(i) $\|P_\Omega(t,s)f\|_q \le C(t-s)^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_p$,

(ii) $\|\mathrm{D}_x P_\Omega(t,s)f\|_p \le C(t-s)^{-\frac{1}{2}}\|f\|_p$

for $(t,s) \in \widetilde{\Lambda}_T$ and $f \in L^p(\Omega)$. Moreover, for $1 < p < q < \infty$ and $f \in L^p(\Omega)$

$$\lim_{t\to s}\left[\|(t-s)^{\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}P_\Omega(t,s)f\|_q + \|(t-s)^{\frac{1}{2}}\mathrm{D}_x P_\Omega(t,s)f\|_p\right] = 0.$$

Proof. To obtain (i) we apply Lemma 3.3 with $X_1 = L^q(\Omega)$, $X_2 = L^p(\Omega)$, $R = W$, $S = F$, $\alpha = -\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)$, $\beta = -\frac{1}{2}$, Proposition 2.3 and Corollary 2.7 in the case where $q \ge p \ge \frac{d}{2}$. By iteration (i) holds also for $1 < p < \frac{d}{2}$.

The second assertion follows by applying Lemma 3.3 with $X_1 = W^{1,p}(\Omega)$, $X_2 = L^p(\Omega)$, $R = W$, $S = F$, $\alpha = \beta = -\frac{1}{2}$, Proposition 2.3 and Corollary 2.7. Finally, the last assertion can be obtained as in [15, Proposition 3.4]. □

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